Wireless Information Flow and Matroid Theory

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Abstract

In this project, we study the combinatorial structures of a recently proposed ADT wireless network model. [3] and [5] prove the mincut-maxflow result in layered ADT model using matroid theory. Based on these works, we extend the result to the *Many to One* multiple access channel and the *One to Many* broadcast channel in ADT network, and derive the capacity regions of these channels.

1 Introduction

In his seminal paper [1], Shannon created information theory and derived the capacity of point to point communication channel. However, characterizing the capacity region in multiuser networks is much more difficult, and the capacity of general relay channel is still an open problem. Recently, Avestimehr, Diggavi and Tse [2] propose a linear deterministic model, which we call ADT model, to study the approximate capacity of unicast and multicast in wireless relay networks. They extend the celebrated mincut-maxflow result in wireline networks to wireless ADT model. In the proof of mincut-maxflow result in ADT model, a random coding technique over long blocks of data is proposed. Later, Sadegh Tabatabaei Yazdi and Savari [3], and Goemans, Iwata and Zenklusen [5] independently study the combinatorial structure of ADT model, and provide an elegant proof of the mincut-maxflow result using matroid theory. They show a flow over the network is sufficient to achieve the unicast capacity of the layered ADT model.

A natural extension of the mincut-maxflow result in unicast session is to study the Multiple Access Channel (MAC) and Broadcast Channel in wireless ADT model. In MAC channel, there are multiple sources, which want to transmit independent messages to one destination simultaneously. The dual of MAC channel is the Broadcast channel, where one source transmits independent messages to multiple destinations. Based on the above unicast mincut-maxflow result, we derive the capacity region of MAC and Broadcast channel in layered wireless ADT model, and show that flow is also sufficient to achieve cut-set bound.

This report is organized as follows. In Section 2 we give a brief overview of the ADT model. Section 3 introduces the mathematical background of linking system and shows how to study the combinatorial structures of flow in ADT networks using matroid theory. In Section 4 we prove the mincut-maxflow result of unicast session in ADT model. In Section 5, we extend the mincut-maxflow result to the MAC and Broadcast channels in ADT network and derive the capacity regions. Section 6 concludes the report and discusses some future work.

2 ADT Model

[2] introduces the ADT model to study the capacity region of relay networks, and they show the capacity region of any wireless Guassian relay channel is within a constant number of bits of the corresponding ADT model. Since the focus of this project is to study the combinatorial structures of ADT model, we give a brief overview of the ADT model and the readers are referred to [2] for more details.

We consider a wireless relay networks with nodes $\mathcal{N} = \{N^1, \ldots, N^q\}$. Every node has a set of $2b$ vertices, where b is an integral constant. For every node N^i in the network, its vertices consist of two parts, the inputs $\{v_1^i, \ldots, v_b^i\}$ and outputs $\{w_1^i, \ldots, w_b^i\}$. We call a wireless network is layered if the nodes can be partitioned into different layers, and all signals from nodes in one layer are sent to outputs of nodes in the next layer. Since very arbitrary directed network can be modeled as a layered network by taking time-expansion ([2]), we only need to consider layered network in this report.

Every time a node receives signals from inputs of nodes in the previous layer and can resend signals to the next layer from its inputs. The communication channel can be characterized by the following equation:

$$w^i = \sum_{\{j|N^j \text{ is in the previous layer of } N^i\}} G_{ji} \cdot v^j,$$

where

- G_{ij} is the channel gain matrix between node N^i and N^j, which is a binary matrix,
- w^i is the outputs of node N^i, and is a binary column vector,
- v^j is the inputs of node N^j, and is a binary column vector,
- the sum is over finite field F_2.

Figure 1 is an example of an ADT model with six nodes, and the constant b is five. The channel gain is determined by the arc between vertices. For

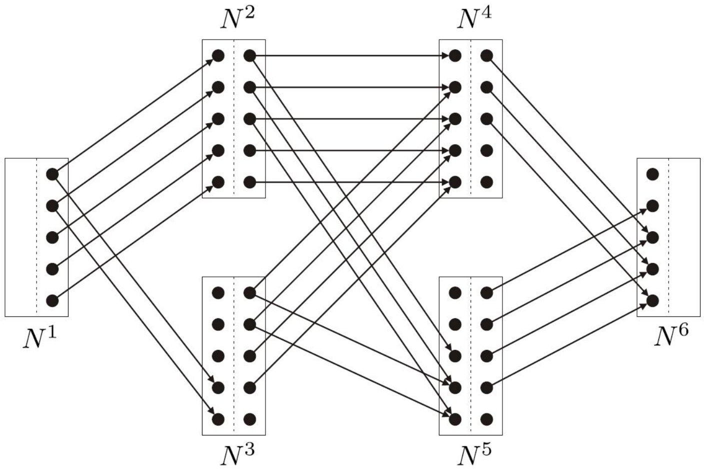

Figure 1: A layered ADT model network with 6 nodes. The source N^1 wants to transmit the information to the destination N^6. (Excerpted from [5].)

example, in this network the channel gain matrix G_{25} is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

From this example it is easy to see the ADT model captures the broadcast nature and interference effects of wireless communication. The question now is what is the capacity region of this network and what is the optimal scheme to transmit maximum information from the source to the destination. Next we will define flow in the ADT network and prove that flow can achieve the capacity in Section 4.

A flow F in the ADT network is a subset of vertices satisfying the following conditions:

1. For every node $N \in \mathcal{N}$, $|F \cap N \cap O| = |F \cap N \cap I|$, i.e., the number of used outputs in each node is equal to the number of used inputs, where O and I are the unions of outputs and inputs of all nodes, respectively.

2. The channel gain matrix M[F] between these used vertices has full rank.

The first condition is essentially the flow conservation and the second condition guarantees that the destination can decode the original information from received signals. The value of a flow F is measured by $|F \cap N^1| = |F \cap N^q|$.

Next we definie an ADT cut in the layered wireless networks. The definition is similar to the one in wireline networks. An ADT cut is a set $C \subseteq V$ such that for every node $N \in \mathcal{N}$, either $N \subseteq C$ or $C \cap N = \emptyset$. Equivalently, an ADT cut is set of of nodes. Further, we require the source $N^1 \in C$ and the destination N^q does not belong to C. We associate every cut C with a value, which is given by

$$\sum_{i=1}^{r-1} \text{rank } (M[(C \cap I_i) \cup (O_{i+1} \backslash C)]).$$

Similar to the case in wireline networks, it is easy to see this cut value provides a upper bound on the value of a maximum ADT flow. Indeed, Avestimehr et al. [2] prove that the minimum value of ADT cut is exactly equal to the maximum value of ADT flow by using a random coding technique. The main result of [3] and [5] is to prove the above result by exploiting the combinatorial structures of ADT model, which we will show in Section 4.

3 Flow Model Based on Matroid Theory and Linking System

In this section, we introduce linking system based on matroid theory and study the combinatorial structure of ADT flow. By using the powerful tool of matroid theory and linking system, we prove the mincut-maxflow result of unicast session in ADT model. For proofs and further details of linking system, readers are referred to [4].

3.1 Preliminaries on Linking Systems

Linking systems were introduced by Schrijver [4]. They map subsets of a ground set to subsets of the other ground set, and preserve some combinatorial structures. In the ADT model, linking systems can be used to describe how information is transmitted from the inputs of one layer to outputs of the next layer. The formal definition of linking systems is given as follows.

A linking system is a triple (V_1, V_2, Λ), where V_1 and V_2 are finite sets, called ground sets, and $\Lambda \subseteq 2^{V_1} \times 2^{V_2}$ such that the following conditions are satisfied:

1. if $(P_1, P_2) \in \Lambda$, then $|P_1| = |P_2|$,
2. if $(P_1, P_2) \in \Lambda$ and $Q_1 \in P_1$, then $\exists Q_2 \subseteq P_2$ with $(Q_1, Q_2) \in \Lambda$,
3. if $(P_1, P_2) \in \Lambda$ and $Q_2 \in P_2$, then $\exists Q_1 \subseteq P_1$ with $(Q_1, Q_2) \in \Lambda$,
4. if $(P_1, P_2), (Q_1, Q_2) \in \Lambda$, then $\exists (R_1, R_2) \in \Lambda$ such that $P_1 \subseteq R_1 \subseteq P_1 \cup Q_1$ and $Q_2 \subseteq R_2 \subseteq P_2 \cup Q_2$.

One important example is linking systems induced by matrices. Let M be a matrix, and V_1 and V_2 denote the rows and columns of M. Let $\Lambda \subseteq 2^{V_1} \times 2^{V_2}$ and $(P_1, P_2) \in \Lambda$ if and only if the submatrix of M generated by rows P_1 and columns P_2 is a nonsingular square matrix.

Similar to the rank function of every matroid, for each linking system there is a corresponding *linking function* $\lambda : 2^{V_1} \times 2^{V_2} \to N^+$, which is defined as

$$\lambda(P_1, P_2) = \max\{|Q_1| | Q_1 \subseteq P_1, Q_2 \subseteq P_2, (Q_1, Q_2) \in \Lambda\}.$$

Therefore, $\lambda(P_1, P_2)$ is the maximum cardinality of an subset of V_1 that can be linked to some subset of V_2. From the definition, it is easy to see a linking function completely determines a linking system. Lemma 1 shows a matroid induced by a linking system.

Lemma 1 *Let (V_1, V_2, Λ) be a linking system with disjoint ground sets. Then the set $\mathcal{B}_\Lambda = \{P_1 \cup (V_2 \backslash P_2) | (P_1, P_2) \in \Lambda\}$ forms the set of bases of a matroid on the ground set $V_1 \cup V_2$.*

Let M_Λ denote the matroid induced by the base sets $\mathcal{B}_\Lambda$, and ρ_Λ be the rank function of M_Λ. The following lemma shows the relation between the linking function λ and the rank function ρ_Λ.

Lemma 2 *Let (V_1, V_2, Λ) be a linking system with disjoint ground sets with linking function λ. For $P_1 \subseteq V_1$ and $P_2 \subseteq V_2$,*

$$\rho_\Lambda(P_1 \cup P_2) = \lambda(P_1, V_2 \backslash P_2) + |P_2|.$$

Next lemma plays a crucial role in proving the mincut-maxflow result and is essentially an application of matroid intersection theorem.

Lemma 3 *Let (V_1, V_2, Λ_1) and (V_2, V_3, Λ_2) be two linking systems, with linking functions λ_1 and λ_2 and define*

$$\Lambda_1 \star \Lambda_2 = \{(P_1, P_3) \in 2^{V_1} \times 2^{V_3} | \exists P_2 \subseteq V_2 : (P_1, P_2) \in \Lambda_1, (P_2, P_3) \in \Lambda_2\}.$$

Then $(V_1, V_3, \Lambda_1 \star \Lambda_2)$ is again a linking system with linking function

$$(\Lambda_1 \star \Lambda_2)(P_1, P_3) = \min_{P_2 \in V_2} \{\lambda_1(P_1, P_2) + \lambda_2(V_2 \backslash P_2, P_3)\}.$$

Indeed, the the linking function of $\Lambda_1 \star \Lambda_2$ can be derived from the matroid intersection theorem by defining two matroids:

$$(M)_1 = (V_2, \mathcal{I}_1), \text{ where } \mathcal{I}_1 = \{P \in V_2 | \lambda_1(P_1, P) = |P|\},$$

$$(M)_2 = (V_2, \mathcal{I}_2), \text{ where } \mathcal{I}_2 = \{P \in V_2 | \lambda_2(P, P_3) = |P|\}.$$

To calculate $\lambda_1 \star \lambda_2(P_1, P_3)$, we need to find the maximum size of a subset $P_2 \subseteq V_2$, such that there exist $Q_1 \subseteq P_1, Q_3 \subseteq P_3$ and $(Q_1, P_2) \in \Lambda_1$, $(P_2, Q_3) \in \Lambda_2$. The set of P_2 satisfying this condition is the exactly the intersection of matroid $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus by matroid intersection theorem,

$$(\Lambda_1 \star \Lambda_2)(P_1, P_3) = \min_{P_2 \in V_2} \{\lambda_1(P_1, P_2) + \lambda_2(V_2 \backslash P_2, P_3)\}.$$

3.2 Linking Networks

In this part, we define linking networks and define flow and cut in linking networks. Let r be an integer, and $G = (V, \Lambda)$, where $V = (V_1, \ldots, V_r)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_{r-1})$. If $(V_i, V_{i+1}, \Lambda_i)$ is a linking system for $1 \leq i \leq r-1$, then we call G a linking network. Further, we call $V_i (1 \leq i \leq r)$ the layers of the linking network G and elements of each layer are called vertices.

A *flow* in a linking network is a tuple $F = (F_1, \ldots, F_r)$, where $F_i \subseteq V_i$ for $1 \leq i \leq r$, and $(F_i, F_{i+1}) \in \Lambda_i$ for $1 \leq i \leq r-1$. The value of flow is defined as the size of F_1, and it is easy to see $|F_1| = |F_i|$ for any i. A $V_1 - V_r$ *cut* in a linking network is a tuple $C = (C_1, \ldots, C_r)$, where $C_i \subseteq V_i$ for $1 \leq i \leq r$, $C_1 = V_1$ and $C_r = \emptyset$. The value of a cut is given by

$$\phi(C) = \sum_{i=1}^{r-1} \lambda_i(C_i, V_{i+1} \backslash C_{i+1}).$$

In Section 4 we will prove the maximum value of flow is exactly the same as the minimum value of cut in linking networks.

3.3 ADT Model and Linking Network

Recall that in layered ADT wireless network we have q layers and every node has $2b$ vertices, which are partitioned into two parts, inputs and outputs. The ADT model can be represented by a linking network with $2q-2$ layers $I_1, Q_2, I_2, \ldots, O_q$. The linking system between I_i and O_{i+1} is induced by the binary channel gain matrix, and the linking system between O_i and I_i is induced by a complete bipartite graph. Figure 2 is an illustration of the ADT model in Figure 1 represented by linking networks.

Before deriving the mincut-maxflow result, we note that the definition of cut in a linking system is slightly different from the definition of ADT cut in Section 2. For an ADT cut, every node is either entirely included in the cut or excluded. But in the definition of cut in linking system, part of vertices of a node can be included in the cut. However, it is easy to show that given any cut C in linking network, we can deduce an ADT cut C' such that the value of C' is not greater than the value of the original cut C, i.e., $\phi(C') \leq \phi(C)$. (see [5] Section III C for detailed proof) This statement implies that the minimum value of ADT cut is the same as the minimum

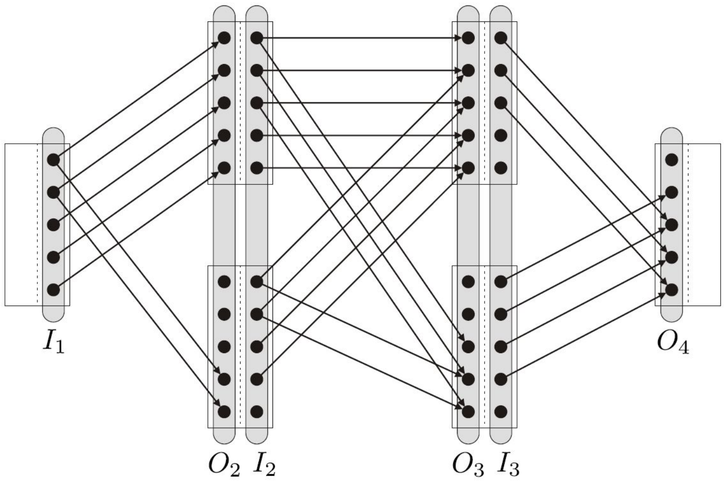

Figure 2: Representation of the ADT model in Figure 1 by linking networks. (Excerpted from [5].)

value of cut in linking network, since ADT cut is a special case of linking network cut.

4 Main Result of Unicast session in ADT Model

The following theorem is the main result of [3] and [5].

Theorem 1 *In any linking network, the value of a maximum flow is equal to the value of a minimum cut.*

Proof: Given a linking network $G = (V, \Lambda)$, where $V = (V_1, \ldots, V_r)$ and $\Lambda = (\Lambda_1, \ldots, \Lambda_{r-1})$. We define a new linking system

$$\Lambda' = V_1 \star V_2 \star \cdots \star V_r.$$

Lemma 3 shows Λ' is indeed a linking system. Let λ' be the linking function of Λ'. By the definition of flow and linking function, the maximum value of flow is $\lambda'(V_1, V_r)$. By Lemma 3 and the definition of cut, we have the following mincut-maxflow result:

$$\begin{aligned}\lambda'(V_1, V_r) &= \min_{P_i \subseteq V_i} \{\lambda_1(V_1, V_2 \backslash P_2) + \sum_{i=2}^{r-2} \lambda_i(P_i, V_{i+1} \backslash P_{i+1}) + \lambda_r(P_{r-1}, V_r)\} \\ &= \min_{P_i \subseteq V_i} \{\phi(V_1 \cup P_2 \cdots \cup P_{r-1})\}\end{aligned}$$

□

Thus in layered ADT model, the capacity region of unicast session is the value of mincut, and a deterministic flow can achieve this capacity. Further, in the max flow, every relay only permutate the received signal and send them out to the next layer without doing any other network coding operations.

5 Our Work: Extensions of Unicast in ADT Model

One natural extension of the previous single-source single-destination model is to study the *Many to One* multiple access channel and the *One to Many* broadcast channel in wireless ADT network model. Based on the work of [3] and [5], We extend the mincut-maxflow result to the multiple access channel and broadcast channel in ADT networks, and characterize the capacity region of these models.

5.1 Multiple Access Channel

First consider the two user multiple access channel, since the same technique can be easily applied to the case of more than two users. Suppose there are two nodes N^1, N^2 in the first layer, and they want to transmit independent information to the same destination N^q with rates $R = (R_1, R_2)$. The question is what is the capacity region of this MAC channel and what is the optimal transmit scheme. It is relatively easy to see the following conditions are necessary for achievable data rates (R_1, R_2):

$$\begin{aligned} R_1 &\leq \min\{\phi(N^1 \cup P_2 \cup \cdots \cup P_{r-1}) | P_i \in V_i, 2 \leq i \leq r-1\} \\ R_2 &\leq \min\{\phi(N^2 \cup P_2 \cup \cdots \cup P_{r-1}) | P_i \in V_i, 2 \leq i \leq r-1\} \\ R_1 + R_2 &\leq \min\{\phi((N^1 \cup N^2) \cup P_2 \cup \cdots \cup P_{r-1}) | P_i \in V_i, 2 \leq i \leq r-1\} \end{aligned}$$

The right hand side of the above inequalities are the mincut value between source and destination, and thus are upper bounds of the achievable rates. By adding a super source node and connecting the super node with each source by orthogonal links, we reduce the MAC channel to single-source and single-destination channel and show that the above conditions are also sufficient.

Given any $R = (R_1, R_2)$ satisfying the above conditions, we create a super source node N^0 with $R_1 + R_2$ inputs, and add R_1 arcs from N^0 to N^1, R_2 arcs from N^0 to N^2, where these arcs are orthogonal. Then we have a single-source single-destination ADT network. Let $\varphi(A, B)$ denote the mincut value between A and B. Consider any ADT cut in this layered network, similar to the case in wireline networks, we have

$$\begin{aligned} \varphi(N^0; N^q) = \min(\; & \varphi(N^0; N^1, N^2, N^q); \varphi(N^0, N^1; N^2, N^q); \\ & \varphi(N^0, N^2; N^1, N^q); \varphi(N^0; N^1, N^2, N^q)) \end{aligned}$$

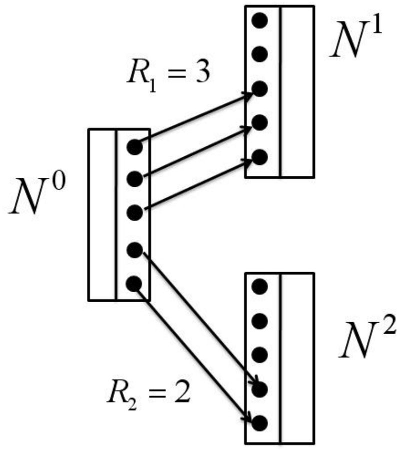

Figure 3: Two user multiple access channel with added super source node.

By looking at the four cuts separately, we can show the mincut between N^0 and N^q is exactly $R_1 + R_2$ (see [6] for detailed proof). So from 4 we know there exists a flow from N^0 to N^q with value of $R_1 + R_2$. Thus all the links from N^0 to N^1 and N^2 are used and information on these links are independent. Thus, N^1 can send data to N^q at a rate of R_1, and N^2 can send data at a rate of R_2. So $R = (R_1, R_2)$ is achievable. Further, during the transmission relay nodes only permutate and resend the signals from the previous layer, without doing any network coding operation. Therefore, we have shown that cut-set conditions are both sufficient and necessary for two user MAC channel in layered ADT wireless network. The same technique can be applied to the case of more than two users.

5.2 Broadcast Channel

In broadcast channel there are multiple destinations in the last layer, and the transmitter wants to send independent information to each destination. We derive the capacity region of the broadcast channel, which is given as follows.

Theorem 2 *Suppose there are r destinations $D = (N^{p1}, \ldots, N^{pr})$ in the last layer, and transmitter N^0 wants to send independent information to each destination. Let $R = (R_1, \ldots, R_r)$ denote data rate from source to each destination, and let $\varphi(A, B)$ denote the mincut value between A and B. The capacity region of broadcast channel is given by*

$$\{R = (R_1, \ldots, R_r) | R(S) \leq \varphi(N^0; S), \forall S \subseteq D\},$$

where $R(S)$ denotes the sum rates from source to subset S of the destinations.

The proof technique is essentially the same as the one in multiple access channel. We first add a super destination node and create orthogonal links from each destination to the super node, and thus reduce the problem to single-source single-destination problem.

6 Conclusion and Future Work

In this project, we study the combinatorial structures of a recently proposed ADT wireless network model. [3] and [5] prove the mincut-maxflow result in layered ADT model using matroid theory. Based on these works, we extend the result to the *Many to One* multiple access channel and the *One to Many* broadcast channel in wireless ADT network model, and derive the capacity region of these channels.

There are still many open problems in wireless networks. In undirected wireline networks, it is known that the cut condition is sufficient for two-commodity flow problem. However, in wireless networks, the capacity region of the simplest two user interference channel is still an open problem and even in linear deterministic ADT model, cut conditions are not sufficient. How to characterize the capacity region of two unicast sessions in layered ADT network is an interesting and important topic.

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